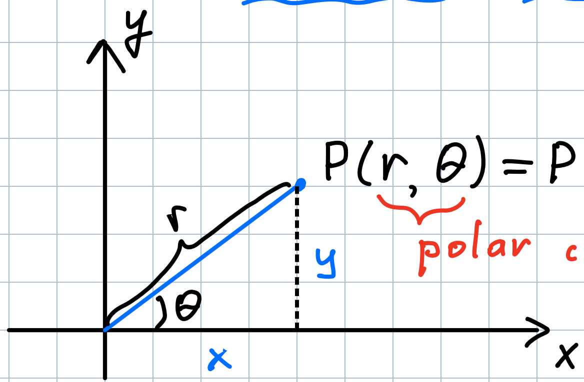
Double integrals in polar coordinates

Last time



$P(r,\theta) = P(x,y)$

$\underbrace{r,\theta}$ polar coordinates

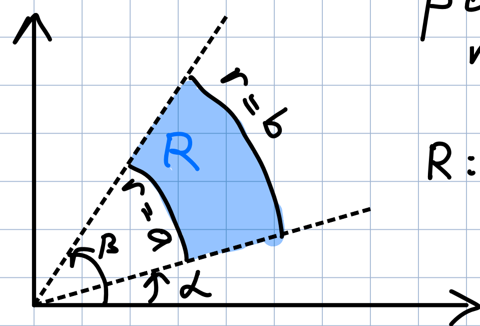
$$x = r\cos\theta$$

$$y = r\sin\theta$$

Assume: $0 \le a \le b$, $0 \le \beta - \alpha \le 2\pi$

don't forget this factor!

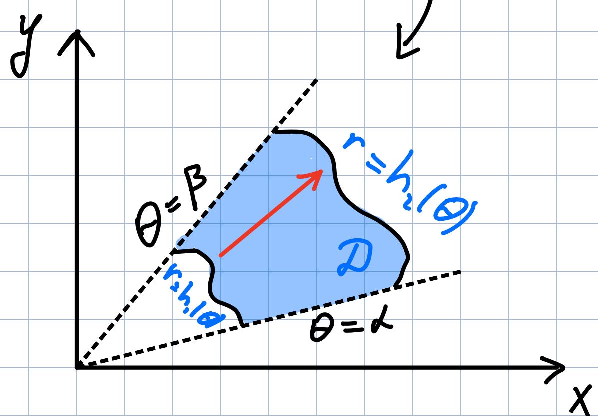
$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$



polar rectangle

$R:\ a \le r \le b$
$\quad\ \alpha \le \theta \le \beta$

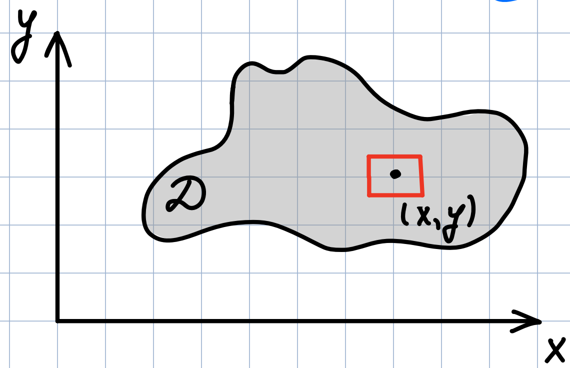
For $\mathcal{D}$ like that



$$\mathcal{D} = \left\{ (r,\theta) \;\middle|\; \begin{array}{l} \alpha \le \theta \le \beta \\ h_1(\theta) \le r \le h_2(\theta) \end{array} \right\}$$

$$\iint_{\mathcal{D}} f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Mass, Center of Mass, Moments



A lamina (plate),

- density $\rho(x,y) = \lim \dfrac{\Delta m}{\Delta A}$

(in units of mass per unit area)

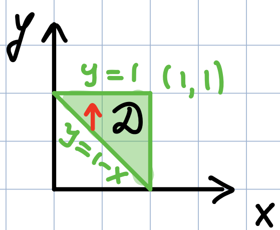
Δm ← mass in a tiny rectangle about (x,y)

ΔA ← area of the rectangle

- total mass

$$m = \lim_{k,\ell\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{\ell} \underbrace{\rho(x_{ij}^*, y_{ij}^*)\,\Delta A}_{\approx \text{ mass in } ij\text{-th rectangle}} = \iint_D \rho(x,y)\,dA$$

<u>Ex:</u>



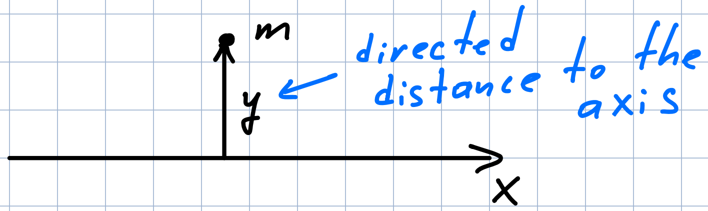
density: $\rho(x,y) = x+y$ $(\mathrm{kg}/\mathrm{m}^2)$.

Find the total mass.

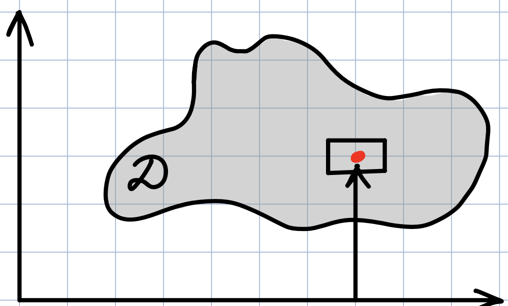
<u>Sol:</u>

$$m = \iint_D \rho(x,y)\,dA = \int_0^1 \underbrace{\int_{1-x}^1 (x+y)\,dy}_{xy+\frac{1}{2}y^2\Big|_{y=1-x}^{y=1} = x+\frac{1}{2}-x(1-x)-\frac{(1-x)^2}{2} = x+\frac{x^2}{2}}\,dx = \int_0^1 \left(x+\frac{1}{2}x^2\right)dx = \frac{x^2}{2}+\frac{x^3}{6}\Big|_{x=0}^{x=1} = \frac{2}{3}$$

Moment of a particle about x-axis: my



Moment of a lamina about x-axis:



$$M_x = \lim \sum_i \sum_j y_{ij}^* \, \rho(x_{ij}^*, y_{ij}^*) \Delta A$$

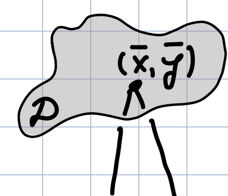
$$= \iint_D y \, \rho(x,y) \, dA$$

Similarly: $M_y = \iint_D x \, \rho(x,y) \, dA$

Center of mass $(\bar{x}, \bar{y})$: $\quad m\bar{x} = M_y \quad , \quad m\bar{y} = M_x$.

So, coordinates of center of mass:

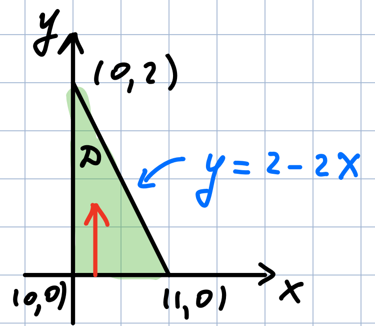
$$\bar{x} = \frac{M_y}{m} = \frac{\iint_D x \, \rho(x,y) \, dA}{\iint_D \rho(x,y) \, dA} \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{\iint_D y \, \rho(x,y) \, dA}{\iint_D \rho(x,y) \, dA}$$



lamina balances when supported at the center of mass

Ex: Find the mass and center of mass of a triangular lamina with vertices $(0,0)$, $(1,0)$, $(0,2)$ and density $\rho(x,y) = y$.

Sol:



$$\mathcal{D}: \quad 0 \le x \le 1, \qquad 0 \le y \le 2-2x$$

$$\boxed{m} = \int_0^1 \underbrace{\int_0^{2-2x} y \, dy}_{\frac{1}{2}y^2 \Big|_{y=0}^{y=2-2x} = 2(1-x)^2} dx = 2\int_0^1 (1-x)^2 \, dx = -2\,\frac{1}{3}(1-x)^3 \Big|_{x=0}^{x=1}$$

$$\boxed{= \frac{2}{3}} \text{ total mass}$$

$$M_y = \int_0^1 \int_0^{2-2x} xy \, dy \, dx = 2\int_0^1 x(1-x)^2 \, dx$$

$$= 2\int_0^1 \left(x^3 - 2x^2 + x\right) dx = 2\left(\frac{x^4}{4} - \frac{2}{3}x^3 + \frac{x^2}{2}\right)\Big|_{x=0}^{x=1}$$

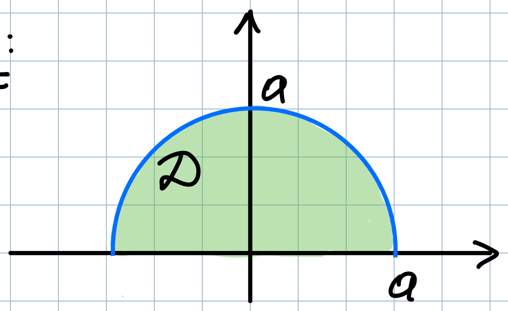
$$= 2\left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right) = \frac{1}{6} \implies \underline{\underline{\bar{x}}} = \frac{M_y}{m} = \frac{1/6}{2/3} = \underline{\underline{\frac{1}{4}}}$$

$$M_x = \int_0^1 \underbrace{\int_0^{2-2x} y^2 \, dy}_{\frac{1}{3}y^3 \Big|_{y=0}^{y=2-2x} = \frac{8}{3}(1-x)^3} dx = \frac{8}{3}\int_0^1 (1-x)^3 \, dx = -\frac{8}{3}\,\frac{(1-x)^4}{4}\Big|_{x=0}^{x=1} = \frac{2}{3}$$

$$\implies \underline{\underline{\bar{y}}} = \frac{M_x}{m} = \frac{2/3}{2/3} = \underline{\underline{1}}$$

Center mass: $\left(\frac{1}{4}, 1\right)$

Ex:



The density at any point on a semicircular lamina is proportional to the distance from the center of the circle.

Find the center of mass.

Semicircular lamina:

$$\begin{cases} x^2+y^2 \le a^2 \\ y \ge 0 \end{cases}$$

$$\rho(x,y) = K\sqrt{x^2+y^2}$$

Sol: $m = \iint_D K\sqrt{x^2+y^2}\, dA \underset{\text{coord.}}{\overset{\text{polar}}{=\!=}} \int_0^\pi \underbrace{\int_0^a Kr \cdot r\, dr}_{\frac{1}{3}Kr^3\Big|_{r=0}^{r=a} = \frac{1}{3}Ka^3} d\theta = \int_0^\pi \frac{1}{3}Ka^3\, d\theta = \frac{\pi}{3}Ka^3$

$M_y = \iint_D K\sqrt{x^2+y^2}\, x\, dA = 0 \qquad \Rightarrow \bar{x} = \frac{M_y}{m} = 0$

By symmetry of D and ρ w.r.t. y-axis

$M_x = \iint_D K\sqrt{x^2+y^2}\, y\, dA \underset{\text{coord.}}{\overset{\text{polar}}{=\!=}} \int_0^\pi \underbrace{\int_0^a Kr^2 \sin\theta\, r\, dr}_{\frac{1}{4}Kr^4\sin\theta\Big|_{r=0}^{r=a} = \frac{1}{4}Ka^4\sin\theta} d\theta = \frac{Ka^4}{4}\int_0^\pi \sin\theta\, d\theta = -\frac{Ka^4}{4}\cos\theta\Big|_{\theta=0}^{\theta=\pi}$

$= \frac{Ka^4}{2}$

$\Rightarrow \bar{y} = \dfrac{\frac{K}{2}a^4}{\frac{\pi}{3}Ka^3} = \frac{3}{2\pi}a$

$\boxed{\text{Center of mass: } \left(0, \frac{3}{2\pi}a\right)}$

Enjoy the Fall Break!

Exam 2

Thu, October 30